Studia Scientiarum Mathematicarum Hungarica 54 (3), 362–393 (2017)
DOI: 10.1556/012.2017.54.3.1371

EXTREME 2-HOMOGENEOUS POLYNOMIALS ON THE PLANE WITH A HEXAGONAL NORM AND APPLICATIONS TO THE POLARIZATION AND UNCONDITIONAL CONSTANTS

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Communicated by A. Kroó

(Received March 6, 2016; accepted March 20, 2017)

Abstract

We classify the extreme 2-homogeneous polynomials on $\mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$. As applications, using its extreme points with the Krein-Milman Theorem, we explicitly compute the polarization and unconditional constants of $\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})$.

1. Introduction

According to the Krein-Milman Theorem, every nonempty convex set in a Banach space is fully described by the set of its extreme points. We recall that if C is a convex set in a Banach space, a point $e \in C$ is said to be *extreme* if $x, y \in C$ and $e = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$ implies that $x = y = e$. Note that if $e \in C$ such that $x, y \in C$ and $e = \frac{1}{2}(x+y)$ implies that $x = y = e$, then e is an extreme point of C. Indeed, without loss of generality, we may assume that $0 < \lambda \leq \frac{1}{2}$. Then, $2\lambda x + (1-2\lambda)y \in C$ and it follows that $e = \lambda x + (1-\lambda)y = \frac{1}{2}[2\lambda x + (1-2\lambda)y] + \frac{1}{2}y$, which shows that

2010 *Mathematics Subject Classification.* Primary 46A22.

Key words and phrases. The Krein-Milman Theorem, extreme points, 2-homogeneous polynomials, the plane with a hexagonal norm, the polarization and unconditional constants.

*This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2057788).

$2\lambda x+(1-2\lambda)y=y$, hence, $x=y$. Let $n\in\mathbb{N}$. We write B_E for the closed unit ball of a real Banach space E. We denote by $extB_E$ the sets of all the extreme points of B_E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\|=\sup_{\|x_k\|=1}|T(x_1,\cdots,x_n)|$. A n-linear form T is symmetric if $T(x_1,\ldots,x_n)=T(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ for every permutation σ on $\{1,2,\ldots,n\}$. We denote by $\mathcal{L}_s(^nE)$ the Banach space of all continuous symmetric n-linear forms on E. A mapping $P:E\to\mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T\in\mathcal{L}_s(^nE)$ such that $P(x)=T(x,\cdots,x)$ for every $x\in E$. In this case it is convenient to write $T=\check{P}$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\|=\sup_{\|x\|=1}|P(x)|$. It is well-known that

$$\|P\|\le\|\check{P}\|\le\frac{n^n}{n!}\|P\|\quad(\forall P\in\mathcal{P}(^nE)).$$

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

In [33], the nth polarization constant of E is defined by

$$c_{\text{pol}}(n:E)=\inf\{M>0:\|\check{P}\|\le M\|P\|\text{ for every }P\in\mathcal{P}(^nE)\}.$$

Let X^α denote the monomial $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$, where $X=(x_1,\ldots,x_m)\in\mathbb{R}^m$ and $\alpha=(\alpha_1,\ldots,\alpha_m)$ with $\alpha_k\in\mathbb{N}\cup\{0\}, 1\le k\le m$. If $P(X)=\sum_{|\alpha|\le n}a_\alpha X^\alpha$ is a polynomial of degree n on $\mathbb{R}^m$, we define its modulus $|P|$ by $|P|(X)=\sum_{|\alpha|\le n}|a_\alpha|X^\alpha$. We define the nth unconditional constant of $\mathbb{R}^m$ by

$$c_{\text{unc}}(n:\mathbb{R}^m)=\inf\{M>0:\||P|\|\le M\|P\|\text{ for every }P\in\mathcal{P}(^n\mathbb{R}^m\}.$$

In 1998, Choi *et al.* [2, 3] characterized the extreme points of the unit ball of $\mathcal{P}(^2l_1^2)$ and $\mathcal{P}(^2l_2^2)$. In 2007, Kim [15] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2l_p^2)$ $(1\le p\le\infty)$. Kim [17, 19, 23] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2=\mathbb{R}^2$ with an octagonal norm $\|(x,y)\|_w=\max\{|x|,|y|,\frac{|x|+|y|}{1+w}\}$. In 2009, Kim [16] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2l_\infty^2)$. Kim [18, 20, 21] also classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2d_*(1,w)^2)$ and $\mathcal{L}(^2d_*(1,w)^2)$. Gamez-Merino *et al.* [8] classified the extreme points of the unit ball of $\mathcal{P}(^2\square)$ and, using its extreme points, compute the polarization and unconditional constants of $\mathcal{P}(^2\square)$, where $\square$ is the unit square of vertices $(0,0),(0,1),(1,0),(1,1)$. We refer to ([1–6], [8–38]) and references therein for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. By the Krein-Milman Theorem, a convex function (like a polynomial norm, for instance) defined on a convex set (like the unit ball of a finite dimensional polynomial space) attains its maximum at one extreme point of the convex set.

We will denote by $P(x,y) = ax^2 + by^2 + cxy$ and $\check{P}((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + \frac{c}{2}(x_1y_2 + x_2y_1)$ a 2-homogeneous polynomial and its corresponding symmetric bilinear form on a real Banach space of dimension 2, respectively. Let $0 < w < 1$ be fixed. We denote by $\mathbb{R}^2_{h(w)}$ the space $\mathbb{R}^2$ endowed with the hexagonal norm

$$\|(x,y)\|_{h(w)} := \max\{|y|,\ |x| + (1-w)|y|\}.$$

Very recently, Kim [24] classified the extreme and exposed points of the unit ball of $\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})$.

In this paper, we classify the extreme points of the unit ball of $\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$. As applications, using its extreme points and the results of [24] with the Krein-Milman Theorem, we explicitly compute $c_{\mathrm{pol}}(2 : \mathbb{R}^2_{h(\frac{1}{2})}) = \frac{5}{4}$ and $c_{\mathrm{unc}}(2 : \mathbb{R}^2_{h(\frac{1}{2})}) = \frac{3}{2}$.

2. The extreme points of the unit ball of $\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$

For $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$, we present an explicit formula of $\|P\|$ in terms of its coefficients a, b, c as follows.

THEOREM 2.1. *Let $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$ with $a \geq 0, c \geq 0$ and $a^2 + b^2 + c^2 \neq 0$. Then:*
Case 1: $c < a$
If $a \leq 4b$, then

$$\begin{aligned}\|P\| &= \max\{a, b, \frac{1}{4}a + b + \frac{1}{2}c, \frac{4ab - c^2}{4a}, \frac{4ab - c^2}{2c + a + 4b}, \frac{4ab - c^2}{-2c + a + 4b}\}\\ &= \max\{a, \frac{1}{4}a + b + \frac{1}{2}c\}.\end{aligned}$$

If $a > 4b$, then $\|P\| = \max\{a, |b|, |\frac{1}{4}a + b| + \frac{1}{2}c, \frac{|c^2 - 4ab|}{4a}\}$.
Case 2: $c \geq a$
If $a \leq 4b$, then $\|P\| = \max\{a, \frac{1}{4}a + b + \frac{1}{2}c, \frac{|c^2 - 4ab|}{2c + a + 4b}\}$.
If $a > 4b$, then $\|P\| = \max\{a, |b|, |\frac{1}{4}a + b| + \frac{1}{2}c, \frac{c^2 - 4ab}{2c - a - 4b}\}$.

Proof. Note that

$$\|P\| = \max\{\max_{0\le x\le \frac{1}{2}} |ax^2 \pm cx + b|, \max_{\frac{1}{2}\le x\le 1} |(a+4b+2c)x^2 - 2(4b+c)x + 4b|,$$
$$\max_{\frac{1}{2}\le x\le 1} |(a+4b-2c)x^2 - 2(4b-c)x + 4b|\}.$$

Let

$$I_1 := \max_{0\le x\le \frac{1}{2}} |ax^2 + cx + b|,$$
$$I_2 := \max_{0\le x\le \frac{1}{2}} |ax^2 - cx + b|,$$
$$J_1 := \max_{\frac{1}{2}\le x\le 1} |(a+4b+2c)x^2 - 2(4b+c)x + 4b|,$$
$$J_2 := \max_{\frac{1}{2}\le x\le 1} |(a+4b-2c)x^2 - 2(4b-c)x + 4b|.$$

Obviously,

$$I_1 = \max\{|b|, |\frac{1}{4}a + b + \frac{1}{2}c|\}.$$

Note that if $c < a$, then

$$I_2 = \max\{|b|, |\frac{1}{4}a + b - \frac{1}{2}c|, \frac{|c^2 - 4ab|}{4|a|}\}$$

and if $c \ge a$, then

$$I_2 = \max\{|b|, |\frac{1}{4}a + b - \frac{1}{2}c|\}.$$

Let's compute J_1. Note that

$$\frac{1}{2} \le \frac{c+4b}{2c+a+4b} \le 1 \text{ if and only if } a \le 4b.$$

Hence, if $a \le 4b$, then

$$J_1 = \max\{|a|, |\frac{1}{4}a + b - \frac{1}{2}c|, \frac{|c^2 - 4ab|}{|2c+a+4b|}\}$$

and if $a > 4b$, then

$$J_1 = \max\{|a|, |\frac{1}{4}a + b - \frac{1}{2}c|\}.$$

Let's compute J_2. Note that

$$\tfrac{1}{2} \leq \tfrac{c-4b}{2c-a-4b} \leq 1 \text{ if and only if } (4b \leq a \leq c, 2c-a-4b \neq 0) \text{ or } (c \leq a \leq 4b, 2c-a-4b \neq 0).$$

Hence, if $(4b \leq a \leq c, 2c-a-4b \neq 0)$ or $(c \leq a \leq 4b, 2c-a-4b \neq 0)$, then

$$J_2 = \max\{|a|, |\frac{1}{4}a + b + \frac{1}{2}c|, \frac{|c^2 - 4ab|}{|2c-a-4b|}\}$$

and otherwise,

$$J_2 = \max\{|a|, |\frac{1}{4}a + b + \frac{1}{2}c|\}.$$

Since $\|P\| = \max\{I_1, I_2, J_1, J_2\}$, it completes the proof. $\square$

Remark. *Note that if* $\|P\| = 1$, *then* $|a| \leq 1, |b| \leq 1, |c| \leq 2$.

We are now in a position to prove the main result of this paper.

Theorem 2.2.

$$\begin{aligned} extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})} = \{&\pm y^2, \pm(x^2 + \tfrac{1}{4}y^2 \pm xy), \pm(x^2 + \tfrac{3}{4}y^2), \\ &\pm[x^2 + (\tfrac{c^2}{4} - 1)y^2 \pm cxy] \ (0 \leq c \leq 1), \\ &\pm[ax^2 + (\tfrac{a+4\sqrt{1-a}}{4} - 1)y^2 \pm (a + 2\sqrt{1-a})xy] \ (0 \leq a \leq 1)\}. \end{aligned}$$

Proof. Let $P(x,y) = ax^2 + by^2 + cxy \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ with $a \geq 0, c \geq 0$. Note that if $b = 1$, then $a = c = 0$. Indeed, since

$$1 = \|P\| \geq |P(\frac{1}{2}, 1)| = \frac{1}{4}a + 1 + \frac{1}{2}c,$$

we have $a = c = 0$. Hence, if $b = 1$, then $P = y^2$.

Claim: $y^2 \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$.

Let

$$Q_1(x,y) = \epsilon x^2 + \gamma y^2 + \delta xy$$

and

$$Q_2(x,y) = -\epsilon x^2 + \gamma y^2 - \delta xy$$

be such that $\|Q_j\| = 1$ and $y^2 = \frac{1}{2}(Q_1 + Q_2)$ for all $j = 1, 2$ and for some $\epsilon, \gamma, \delta \in \mathbb{R}$. Obviously, $\gamma = 1$. Without loss of generality, we may assume that $\delta \geq 0$. If $\epsilon < 0$, then

$$1 = \|Q_2\| \geq |Q_2(\frac{1}{2}, -1)| = \frac{1}{4}|\epsilon| + 1 + \frac{1}{2}\delta > 1,$$

which is a contradiction. Therefore, $\epsilon \geq 0$. Since

$$1 = \|Q_1\| \geq |Q_1(\frac{1}{2}, 1)| = \frac{1}{4}\epsilon + 1 + \frac{1}{2}\delta,$$

we have $\epsilon = 0 = \delta$. Therefore, $y^2 = Q_1 = Q_2$. Hence, $y^2 \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$.

Suppose that $-1 \leq b < 1$.

(Case 1): $a = 1, b = -1$.

We claim that the only extreme point of the unit ball in this case is $P = x^2 - y^2$. Since

$$1 = \|P\| \geq |P(\frac{1}{2}, -1)| = \frac{3}{4} + \frac{1}{2}c,$$

we have $c \leq \frac{1}{2}$. By Theorem 2.1 (case 1),

$$1 = \|P\| \geq \frac{c^2 + 4}{4},$$

which shows that $c = 0$. Hence, if $a = 1, b = -1$, then $P = x^2 - y^2$.

Let

$$Q_1(x, y) = x^2 - y^2 + \delta xy$$

and

$$Q_2(x, y) = x^2 - y^2 - \delta xy$$

be such that $\|Q_j\| = 1$ and $x^2 - y^2 = \frac{1}{2}(Q_1 + Q_2)$ for all $j = 1, 2$ and for some $\delta \geq 0$. Since

$$1 = \|Q_1\| \geq |Q_1(\frac{1}{2}, -1)| = \frac{3}{4} + \frac{1}{2}\delta,$$

we have $\delta \leq \frac{1}{2}$. By Theorem 2.1 (case 1),

$$1 = \|Q_1\| \geq \frac{4 + \delta^2}{4},$$

which implies that $\delta = 0$.

(Case 2): $a = 1$ and $-1 < b < 1$.

We claim that the only extreme point of the unit ball in this case is

$$P = x^2 + \frac{1}{4}y^2 + xy \text{ or } P = x^2 + \frac{3}{4}y^2 \text{ or}$$
$$P = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \text{ for } 0 < c \leq 1.$$

First, assume that $c > 1$. If $1 \leq 4b$, then

$$1 = \|P\| \geq |P(\frac{1}{2}, 1)| = \frac{1}{4} + b + \frac{1}{2}c > 1,$$

which is impossible. Hence, $4b < 1$. By Theorem 2.1 (case 2),

$$1 = \|P\| = \max\{1, |b|, |\frac{1}{4} + b| + \frac{1}{2}c, \frac{c^2 - 4b}{2c - 1 - 4b}\}.$$

Since

$$\frac{c^2 - 4b}{2c - 1 - 4b} \leq 1,$$

we have $c = 1$, which is contradiction. Therefore, $c \leq 1$. If $1 \leq 4b$, by Theorem 2.1 (case 1),

$$1 = \|P\| = \max\{1, b, \frac{1}{4} + b + \frac{1}{2}c\}.$$

Hence,

$$\frac{1}{4} \leq b \leq \frac{3}{4}.$$

We will show that

$$\frac{1}{4} + b + \frac{1}{2}c = 1.$$

Assume that $\frac{1}{4} + b + \frac{1}{2}c < 1$. If $\frac{1}{4} < b \leq \frac{3}{4}$, we define

$$Q_1(x, y) = x^2 + (b + \frac{1}{n})y^2 + (c - \frac{2}{n})xy$$

and

$$Q_2(x, y) = x^2 + (b - \frac{1}{n})y^2 + (c + \frac{2}{n})xy,$$

where

$$0 < b - \frac{1}{n} < b + \frac{1}{n} < 1, 0 < c - \frac{2}{n} < c + \frac{2}{n} < 1 \quad \text{for some } n \in \mathbb{N}.$$

By Theorem 2.1, $\|Q_j\| = 1$ for $j = 1, 2$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which shows that P is not extreme, reducing thus a contradiction. If $b = \frac{1}{4}$, we define

$$Q_1(x, y) = x^2 + by^2 + (c - \frac{2}{n})xy$$

and

$$Q_2(x, y) = x^2 + by^2 + (c + \frac{2}{n})xy,$$

where

$$0 < c - \frac{2}{n} < c + \frac{2}{n} < 1, \frac{1}{4} + b + \frac{1}{2}c + \frac{1}{n} < 1 \text{ for some } n \in \mathbb{N}.$$

By Theorem 2.1, $\|Q_j\| = 1$ for $j = 1, 2$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which shows that P is not extreme, reducing thus a contradiction. Therefore,

$$\frac{1}{4} + b + \frac{1}{2}c = 1.$$

We have shown that

$$P = x^2 + by^2 + (\frac{3}{2} - 2b)xy \quad \text{for } \frac{1}{4} \leq b \leq \frac{3}{4}.$$

Note that $b = \frac{1}{4}$ or $\frac{3}{4}$. Indeed, suppose that

$$\frac{1}{4} < b < \frac{3}{4}.$$

Let $\frac{1}{4} < b_1 < b < b_2 < \frac{3}{4}$ be such that $b = \frac{1}{2}(b_1 + b_2)$. Let

$$Q_j = x^2 + b_j y^2 + (\frac{3}{2} - 2b_j)xy \quad \text{for} \quad j = 1, 2.$$

By Theorem 2.1, $\|Q_j\| = 1$ for $j = 1, 2$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which shows that P is not extreme. Hence, if $1 \leq 4b$, then

$$P = x^2 + \frac{1}{4}y^2 + xy \quad \text{or} \quad P = x^2 + \frac{3}{4}y^2.$$

Claim: $x^2+\frac{1}{4}y^2+xy\in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$.

Let

$$Q_1(x,y)=x^2+(\frac{1}{4}+\epsilon)y^2+(1+\delta)xy$$

and

$$Q_2(x,y)=x^2+(\frac{1}{4}-\epsilon)y^2+(1-\delta)xy$$

be such that $\|Q_j\|=1$ for all $j=1,2$ and for some $\epsilon,\delta\in\mathbb{R}$. Without loss of generality, we may assume that $\delta\geq 0$. If $\epsilon>0$, then

$$1=\|Q_1\|\geq|Q_1(\frac{1}{2},1)|=|\frac{1}{4}+(\frac{1}{4}+\epsilon)+\frac{1}{2}(1+\delta)|>1,$$

which is a contradiction. Hence,

$$\epsilon\leq 0.$$

By Theorem 2.1 (case 2),

$$1=\|Q_1\|\geq\frac{4(\frac{1}{4}+\epsilon)-(1+\delta)^2}{1+4(\frac{1}{4}+\epsilon)-2(1+\delta)},$$

which implies that $\delta=0$. Since

$$1=\|Q_2\|=|Q_2(\frac{1}{2},1)|\geq|\frac{1}{4}+(\frac{1}{4}-\epsilon)+\frac{1}{2}|=1+|\epsilon|,$$

which implies $\epsilon=0$.

Claim: $x^2+\frac{3}{4}y^2\in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$.

Let

$$Q_1(x,y)=x^2+(\frac{3}{4}+\epsilon)y^2+\delta xy$$

and

$$Q_2(x,y)=x^2+(\frac{3}{4}-\epsilon)y^2-\delta xy$$

be such that $\|Q_j\|=1$ for all $j=1,2$ and for some $\epsilon,\delta\in\mathbb{R}$. Without loss of generality, we may assume that $\delta\geq 0$. Since $|Q_j(\frac{1}{2},1)|\leq 1$ for $j=1,2$, we have

$$\epsilon=-\frac{1}{2}\delta.$$

Let

$$Q_3(x,y) = x^2 + (\frac{3}{4} + \frac{1}{2}\delta)y^2 + \delta xy.$$

It follows that

$$\|Q_3\| = \sup_{\|(x,y)\|_{h(\frac{1}{2})}=1} |Q_3(x,-y)| = \sup_{\|(x,y)\|_{h(\frac{1}{2})}=1} |Q_2(x,y)| = \|Q_2\|.$$

Since

$$1 = \|Q_2\| = \|Q_3\| \geq |Q_3(\frac{1}{2},1)| = 1+\delta,$$

we have $\delta = 0 = \epsilon$.

Suppose that $4b < 1$. By Theorem 2.1 (case 1),

$$1 = \|P\| = \max\{1, |b|, |\frac{1}{4} + b| + \frac{1}{2}c, \frac{|c^2 - 4b|}{4}\}.$$

Using the fact that P is extreme, we will show that

$$c > 0 \ \text{ and } \ \frac{|c^2-4b|}{4} = 1.$$

Otherwise.

$$c = 0 \text{ or } \ \frac{|c^2-4b|}{4} < 1.$$

If $c = 0$ then

$$P = x^2 + by^2 \ \text{ for } \ -1 < b < \frac{1}{4},$$

which is not extreme. That is a contradiction. Hence $c > 0$. Assume that $\frac{|c^2-4b|}{4} < 1$. If $c = 1$, then

$$P = x^2 + by^2 + xy \text{ for } -\frac{3}{4} < b < \frac{1}{4},$$

which is not extreme. That is a contradiction. Therefore,

$$0 < c < 1 \text{ and } \ \frac{1}{4} + b + \frac{1}{2}c < 1.$$

Let $n \in \mathbb{N}$ such that

$$b + \frac{1}{n} < \frac{1}{4}, 0 < c - \frac{2}{n} < c + \frac{2}{n} < 1, \frac{1}{4} + b + \frac{1}{2}c + \frac{1}{n} < 1,$$

$$\frac{|c^2 - 4b + \frac{4}{n^2}(1 \pm 3n)|}{4} < 1.$$

Let

$$Q_1(x,y) = x^2 + (b + \frac{1}{n})y^2 + (c + \frac{2}{n})xy$$

and

$$Q_2(x,y) = x^2 + (b - \frac{1}{n})y^2 + (c - \frac{2}{n})xy.$$

By Theorem 2.1 (case 1), $\|Q_j\| = 1$ for $j = 1, 2$. Since $P = \frac{1}{2}(Q_1 + Q_2)$, P is not extreme, which is a contradiction. Hence,

$$\frac{|c^2 - 4b|}{4} = 1.$$

Therefore, if $4b < 1$, then

$$P = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \quad \text{for} \quad 0 < c \leq 1.$$

Claim: $x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ for $0 < c \leq 1$.

Let $0 < c < 1$ and let

$$Q_1(x,y) = x^2 + (\frac{c^2}{4} - 1 + \delta)y^2 + (c + \gamma)xy$$

and

$$Q_2(x,y) = x^2 + (\frac{c^2}{4} - 1 - \delta)y^2 + (c - \gamma)xy$$

be such that $\|Q_j\| = 1$ for all $j = 1, 2$ and for some $\delta, \gamma \in \mathbb{R}$. Since

$$|\frac{c^2}{4} - 1 \pm \delta| \leq 1,$$

we have

$$|\delta| < \frac{1}{4} \quad \text{and} \quad -1 \leq \frac{c^2}{4} - 1 - |\delta| \leq \frac{c^2}{4} - 1 + |\delta| < -\frac{1}{2}.$$

Without loss of generality, we may assume that $\gamma \geq 0$. We will show that

$$c + \gamma < 1.$$

Assume that $1 \leq c+\gamma \leq 2$. By Theorem 2.1 (case 2),

$$1 = \|Q_1\| \geq \frac{(c+\gamma)^2 - 4(\frac{c^2}{4} - 1 + \delta)}{2(c+\gamma) - 1 - 4(\frac{c^2}{4} - 1 + \delta)},$$

which implies that $c+\gamma = 1$. Hence,

$$Q_1(x,y) = x^2 + (\frac{c^2}{4} - 1 + \delta)y^2 + xy$$

and

$$Q_2(x,y) = x^2 + (\frac{c^2}{4} - 1 - \delta)y^2 + (2c-1)xy.$$

By Theorem 2.1 (case 1), it follows that, for $j = 1, 2$,

$$\begin{aligned} 1 = \|Q_j\| &\geq |\frac{1}{4} + (\frac{c^2}{4} - 1 \pm \delta)| + \frac{1}{2} \\ &= -(\frac{1}{4} + (\frac{c^2}{4} - 1 \pm \delta)) + \frac{1}{2}, \end{aligned}$$

which shows that

$$\frac{3 - c^2}{4} \pm \delta \leq \frac{1}{2}.$$

Hence,

$$c^2 \geq 1,$$

which is a contradiction because $0 < c < 1$. Therefore,

$$c + \gamma < 1.$$

By Theorem 2.1 (case 1),

$$1 = \|Q_1\| \geq \frac{4 + 2c\gamma + \gamma^2 - 4\delta}{4},$$

which implies that

$$(*) \qquad 4 + 2c\gamma + \gamma^2 - 4\delta \leq 4.$$

Since

$$|c - \gamma| \leq c + \gamma < 1,$$

by Theorem 2.1 (case 1),

$$1 = \|Q_2\| = \|x^2 + (\frac{c^2}{4} - 1 - \delta)y^2 + (c-\gamma)xy\| \geq \frac{4 - 2c\gamma + \gamma^2 + 4\delta}{4},$$

which implies that

$$(**) \qquad 4 - 2c\gamma + \gamma^2 + 4\delta \leq 4.$$

Adding $(*)$ and $(**)$, we have $8 + 2\gamma^2 \leq 8$, hence, $\gamma = 0$. By $(*)$ and $(**)$, we have

$$4 - 4\delta \leq 4, \; 4 + 4\delta \leq 4,$$

so $\delta = 0$. Therefore,

$$x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})} \text{ for } 0 < c < 1.$$

We will show that if $c = 1$, then

$$P = x^2 - \frac{3}{4}y^2 + xy$$

is extreme. Let

$$Q_1(x,y) = x^2 + (-\frac{3}{4} + \epsilon)y^2 + (1+\delta)xy$$

and

$$Q_2(x,y) = x^2 + (-\frac{3}{4} - \epsilon)y^2 + (1-\delta)xy$$

be such that $\|Q_j\| = 1$ for all $j = 1, 2$ and for some $\epsilon, \delta \in \mathbb{R}$. Since

$$|-\frac{3}{4} \pm \epsilon| \leq 1,$$

we have

$$|\epsilon| \leq \frac{1}{4}.$$

Hence,

$$-1 \leq -\frac{3}{4} - |\epsilon| \leq -\frac{3}{4} + |\epsilon| \leq -\frac{1}{2}.$$

Without loss of generality, we may assume that $\delta \geq 0$. By Theorem 2.1 (case 2),

$$1 = \|Q_1\| \geq \frac{4 + 2\delta - 4\epsilon + \delta^2}{4 + 2\delta - 4\epsilon},$$

which implies that $\delta = 0$. Since, for $j = 1, 2$,

$$1 = \|Q_j\| \geq |\frac{1}{4} + (-\frac{3}{4} \pm \epsilon)| + \frac{1}{2} = 1 + |\epsilon|,$$

which shows that $\epsilon = 0$. Hence,

$$P = x^2 - \frac{3}{4}y^2 + xy \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}.$$

(Case 3): $0 \leq a < 1$ and $b = -1$.
If $c \geq a$, then

$$1 = \|P\| \geq |\frac{1}{4}a - 1| + \frac{1}{2}c,$$

which shows that

$$c \leq \frac{1}{2}a.$$

Hence, $a = 0 = c$ and $P = -y^2$, which is extreme. Suppose that $c < a$. By Theorem 2.1 (case 1),

$$1 = \|P\| = \max\{a, 1, |\frac{1}{4}a - 1| + \frac{1}{2}c, \frac{c^2 + 4a}{4a}\}.$$

Hence

$$\frac{c^2 + 4a}{4a} \leq 1,$$

which implies that $c = 0$. Hence

$$P = ax^2 - y^2 \ \text{ for } \ 0 < a < 1,$$

which is a contradiction because P is extreme. Indeed, let $0 < a_1 < a < a_2 < 1$ be such that $a = \frac{1}{2}(a_1 + a_2)$. Define

$$Q_j(x, y) = a_j x^2 - y^2 \ \text{ for } \ j = 1, 2.$$

Then $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme.
(Case 4): $0 \leq a < 1$ and $-1 < b < 1$

If $-1 < b \leq 0$, we claim that the only extreme point of the unit ball is

$$P = 2xy.$$

Assume that $c < a$. By Theorem 2.1 (case 1),

$$1 = \|P\| = \max\{a, |b|, |\frac{1}{4}a + b| + \frac{1}{2}c, \frac{c^2 - 4ab}{4a}\}.$$

Since P is extreme, we claim that

$$1 = |\frac{1}{4}a + b| + \frac{1}{2}c = \frac{c^2 - 4ab}{4a}.$$

Assume that

$$(|\frac{1}{4}a + b| + \frac{1}{2}c = 1, \frac{c^2 - 4ab}{4a} < 1) \text{ or } (|\frac{1}{4}a + b| + \frac{1}{2}c < 1, \frac{c^2 - 4ab}{4a} = 1).$$

We will derive a contradiction. Let

$$|\frac{1}{4}a + b| + \frac{1}{2}c = 1, \frac{c^2 - 4ab}{4a} < 1.$$

Note that

$$0 < c < 1 \text{ and } \frac{1}{4}a + b < 0.$$

Let $n \in \mathbb{N}$ be such that

$$\begin{aligned} &\frac{1}{4}a + b + \frac{5}{4n} < 0, 0 < c - \frac{5}{2n} < c + \frac{5}{2n} < a - \frac{1}{n} < a + \frac{1}{n} < 1, \\ &-1 < b - \frac{1}{n} < b + \frac{1}{n} < 1, a - \frac{1}{n} > 4(b + \frac{1}{n}), \\ &\frac{(c \pm \frac{5}{2n})^2 - 4(a \pm \frac{1}{n})(b \pm \frac{1}{n})}{4(a \pm \frac{1}{n})} < 1. \end{aligned}$$

Let

$$Q_1(x, y) = (a + \frac{1}{n})x^2 + (b + \frac{1}{n})y^2 + (c + \frac{5}{2n})xy$$

and

$$Q_2(x, y) = (a - \frac{1}{n})x^2 + (b - \frac{1}{n})y^2 + (c - \frac{5}{2n})xy.$$

By Theorem 2.1 (case 1), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction.

Let next

$$|\frac{1}{4}a + b| + \frac{1}{2}c < 1, \frac{c^2 - 4ab}{4a} = 1.$$

Then $c > 0$. Let $\epsilon > 0$ be such that

$$\begin{aligned}
&0 < a - \epsilon < a + \epsilon < 1, -1 < b - (\frac{1+b}{a})\epsilon < b + (\frac{1+b}{a})\epsilon < 1, \\
&0 < c - \frac{4(1+b)}{c}\epsilon < c + \frac{4(1+b)}{c}\epsilon < a - \epsilon, 4(b + (\frac{1+b}{a})\epsilon) < a - \epsilon, \\
&|\frac{1}{4}((a \pm \epsilon) + (b \pm (\frac{1+b}{a})\epsilon)| + \frac{1}{2}(c \pm \frac{4(1+b)}{c}\epsilon) < 1.
\end{aligned}$$

Let

$$Q_1(x, y) = (a + \epsilon)x^2 + (b + (\frac{1+b}{a})\epsilon)y^2 + (c + \frac{4(1+b)}{c}\epsilon)xy$$

and

$$Q_2(x, y) = (a - \epsilon)x^2 + (b - (\frac{1+b}{a})\epsilon)y^2 + (c - \frac{4(1+b)}{c}\epsilon)xy.$$

From the fact that

$$\frac{c^2 - 4ab}{4a} = 1,$$

we deduce that

$$\begin{aligned}
&\frac{(c + \frac{4(1+b)}{c}\epsilon)^2 - 4(a + \epsilon)(b + (\frac{1+b}{a})\epsilon)}{4(a + \epsilon)} = 1 \\
&= \frac{(c - \frac{4(1+b)}{c}\epsilon)^2 - 4(a - \epsilon)(b - (\frac{1+b}{a})\epsilon)}{4(a - \epsilon)},
\end{aligned}$$

hence, by Theorem 2.1 (case 1), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Therefore, we should have

$$1 = |\frac{1}{4}a + b| + \frac{1}{2}c = \frac{c^2 - 4ab}{4a}.$$

Hence, $\frac{1}{4}a + b < 0$ and $c = a$, which is a contradiction. Therefore, we have

$$c \geq a.$$

By Theorem 2.1 (case 2),

$$1 = \|P\| = \max\{a, |b|, |\frac{1}{4}a + b| + \frac{1}{2}c, \frac{c^2 - 4ab}{2c - a - 4b}\}.$$

Since P is extreme, we claim that

$$(***) \qquad 1 = |\frac{1}{4}a + b| + \frac{1}{2}c = \frac{c^2 - 4ab}{2c - a - 4b}.$$

Assume that

$$(|\frac{1}{4}a+b| + \frac{1}{2}c = 1, \frac{c^2 - 4ab}{2c - a - 4b} < 1) \text{ or } (|\frac{1}{4}a+b| + \frac{1}{2}c < 1, \frac{c^2 - 4ab}{2c - a - 4b} = 1).$$

We will derive a contradiction. Let

$$|\frac{1}{4}a + b| + \frac{1}{2}c = 1, \frac{c^2 - 4ab}{2c - a - 4b} < 1.$$

Suppose that $c = a$. Then

$$\frac{1}{4}a + b < 0 \text{ and } P = ax^2 + (\frac{1}{4}a - 1)y^2 + axy \text{ for } 0 < a < 1.$$

We claim that such P is not extreme. Indeed, let $n \in \mathbb{N}$ be such that

$$0 < a - \frac{1}{n} < a + \frac{1}{n} < 1, \ -1 < \frac{1}{4}(a - \frac{1}{n}) - 1 < \frac{1}{4}(a + \frac{1}{n}) - 1 < 0.$$

Let

$$Q_1(x, y) = (a + \frac{1}{n})x^2 + (\frac{1}{4}(a + \frac{1}{n}) - 1)y^2 + (a + \frac{1}{n})xy$$

and

$$Q_2(x, y) = (a - \frac{1}{n})x^2 + (\frac{1}{4}(a - \frac{1}{n}) - 1)y^2 + (a - \frac{1}{n})xy.$$

By Theorem 2.1 (case 2), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Hence, $c > a$. Note that

$$|\frac{1}{4}a + b| > 0.$$

Indeed, if $|\frac{1}{4}a+b|=0$, then

$$\frac{c^2-4ab}{2c-a-4b}=1,$$

which we assumed did not hold. Note that

$$c>0 \text{ and } a>0.$$

If $a=0$, then

$$P=by^2+2(1+b)xy \text{ for } -1<b<0.$$

We claim that such P is not extreme. Indeed, let $n\in\mathbb{N}$ be such that

$$-1<b-\frac{1}{n}<b+\frac{1}{n}<0,\ 0<2(1+b-\frac{1}{n})<2(1+b+\frac{1}{n})<2.$$

Let

$$Q_1(x,y)=(b+\frac{1}{n})y^2+2(1+b+\frac{1}{n})xy$$

and

$$Q_2(x,y)=(b-\frac{1}{n})y^2+2(1+b-\frac{1}{n})xy.$$

By Theorem 2.1 (case 2), $\|Q_j\|=1$ and $P=\frac{1}{2}(Q_1+Q_2)$, which implies that P is not extreme, reaching thus a contradiction.

First, suppose that

$$\frac{1}{4}a+b<0.$$

Let $n\in\mathbb{N}$ be such that

$$\begin{aligned}
&\frac{1}{4}a+b+\frac{5}{4n}<0, 0<a-\frac{1}{n}<a+\frac{1}{n}<1, a+\frac{1}{n}<c-\frac{5}{2n},\\
&-1<b-\frac{1}{n}<b+\frac{1}{n}<1, a-\frac{1}{n}>4(b+\frac{1}{n}),\\
&\frac{(c\pm\frac{5}{2n})^2-4(a\pm\frac{1}{n})(b\pm\frac{1}{n})}{2(c\pm\frac{5}{2n})-(a\pm\frac{1}{n})-4(b\pm\frac{1}{n})}<1.
\end{aligned}$$

Let

$$Q_1(x,y)=(a+\frac{1}{n})x^2+(b+\frac{1}{n})y^2+(c+\frac{5}{2n})xy$$

and

$$Q_2(x,y) = (a - \frac{1}{n})x^2 + (b - \frac{1}{n})y^2 + (c - \frac{5}{2n})xy.$$

By Theorem 2.1 (case 2), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Next, suppose that

$$\frac{1}{4}a + b > 0.$$

Let $n \in \mathbb{N}$ be such that

$$\begin{aligned}
&\frac{1}{4}a + b - \frac{5}{4n} > 0, 0 < a - \frac{1}{n} < a + \frac{1}{n} < 1, a + \frac{1}{n} < c - \frac{5}{2n},\\
&-1 < b - \frac{1}{n} < b + \frac{1}{n} < 1, a - \frac{1}{n} > 4(b + \frac{1}{n}),\\
&\frac{(c \mp \frac{5}{2n})^2 - 4(a \pm \frac{1}{n})(b \pm \frac{1}{n})}{2(c \pm \frac{5}{2n}) - (a \pm \frac{1}{n}) - 4(b \pm \frac{1}{n})} < 1.
\end{aligned}$$

Let

$$Q_1(x,y) = (a + \frac{1}{n})x^2 + (b + \frac{1}{n})y^2 + (c - \frac{5}{2n})xy$$

and

$$Q_2(x,y) = (a - \frac{1}{n})x^2 + (b - \frac{1}{n})y^2 + (c + \frac{5}{2n})xy.$$

By Theorem 2.1 (case 2), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction.

Let us show next that we can not have

$$|\frac{1}{4}a + b| + \frac{1}{2}c < 1, \frac{c^2 - 4ab}{2c - a - 4b} = 1.$$

Note that in that case, we have

$$c > a > 0.$$

Indeed, if $c = a$, then $a = 1$, which is impossible and if $a = 0$, then

$$2 > c = 1 + \sqrt{1 + 4|b|} \geq 2,$$

which is also impossible. Let $\epsilon > 0$ be such that

$$0 < a - \epsilon < a + \epsilon < 1, -1 < b - \frac{1 - 4b}{4(1-a)}\epsilon < b + \frac{1 - 4b}{4(1-a)}\epsilon < 1,$$

$$a+\epsilon < c-(\frac{1-4b}{1-c})\epsilon < c+(\frac{1-4b}{1-c})\epsilon < 2, 4(b+\frac{1-4b}{4(1-a)}\epsilon) < a-\epsilon,$$
$$|\frac{1}{4}(a\pm\epsilon)+(b\pm\frac{1-4b}{4(1-a)}\epsilon)|+\frac{1}{2}(c\pm(\frac{1-4b}{1-c})\epsilon) < 1.$$

Let

$$Q_1(x,y)=(a+\epsilon)x^2+(b+\frac{1-4b}{4(1-a)}\epsilon)y^2+(c+(\frac{1-4b}{1-c})\epsilon)xy$$

and

$$Q_2(x,y)=(a-\epsilon)x^2+(b-\frac{1-4b}{4(1-a)}\epsilon)y^2+(c-(\frac{1-4b}{1-c})\epsilon)xy.$$

By Theorem 2.1 (case 2), $\|Q_j\|=1$ and $P=\frac{1}{2}(Q_1+Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Therefore, our claim that

$$|\frac{1}{4}a+b|+\frac{1}{2}c=\frac{c^2-4ab}{2c-a-4b}=1$$

is proved.

If $\frac{1}{4}a+b\geq 0$, then

$$c=2-a \ \text{ and } \ b=\frac{a}{4}\geq 0,$$

which show that

$$0=a=b, c=2.$$

Claim: $2xy\in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$

Let

$$Q_1(x,y)=\epsilon x^2+\delta y^2+2xy$$

and

$$Q_2(x,y)=-\epsilon x^2-\delta y^2+2xy$$

be such that $\|Q_j\|=1$ for all $j=1,2$ and for some $\epsilon\geq 0$ and $\delta\in\mathbb{R}$. Note that

$$1=\|Q_1\|\geq|Q_1(1,0)|=\epsilon.$$

Since

$$1=\max\{\|Q_1\|,\|Q_2\|\}\geq\max\{|Q_1(\frac{1}{2},1)|,|Q_2(\frac{1}{2},1)|\}=|\frac{1}{4}\epsilon+\delta|+1,$$

which follows that $\delta = -\frac{1}{4}\epsilon$. Again, by Theorem 2.1 (case 2),

$$1 = \|Q_1\| \geq \frac{4+\epsilon^2}{4},$$

hence, $\epsilon = 0 = \delta$.

If $\frac{1}{4}a + b < 0$, by a calculation,

$$b = \frac{a + 4\sqrt{1-a}}{4} - 1 \text{ and } \ c = a + 2\sqrt{1-a} \text{ for } 0 < a < 1.$$

Hence,

$$P = ax^2 + (\frac{a+4\sqrt{1-a}}{4} - 1)y^2 + (a + 2\sqrt{1-a})xy \ \text{ for } \ 0 < a < 1.$$

Claim: $P = ax^2 + (\frac{a+4\sqrt{1-a}}{4} - 1)y^2 + (a+2\sqrt{1-a})xy \in extB_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ for $0 < a < 1$ Let

$$b = \frac{a + 4\sqrt{1-a}}{4} - 1, c = a + 2\sqrt{1-a} \ \text{ for } \ 0 < a < 1,$$

and

$$Q_1(x,y) = (a+\epsilon)x^2 + (b+\delta)y^2 + (c+\gamma)xy,$$
$$Q_2(x,y) = (a-\epsilon)x^2 + (b-\delta)y^2 + (c-\gamma)xy$$

be such that $\|Q_j\| = 1$ for all $j = 1, 2$ and for some $\epsilon, \delta, \gamma \in \mathbb{R}$. Without loss of generality, we may assume that $\gamma \geq 0$. Note that

$$4b < 0 < a < 1 \leq c.$$

By Remark,

$$\gamma \leq 2 - c.$$

Hence

$$c - \gamma > 0.$$

Since

$$\|(\frac{1}{2}, -1)\|_{h(\frac{1}{2})} = \|(\frac{c-4b}{2c-a-4b}, \frac{2c-2a}{2c-a-4b})\|_{h(\frac{1}{2})} = 1,$$

we have

$$|Q_j(\frac{1}{2}, -1)| \leq 1, |Q_j(\frac{c-4b}{2c-a-4b}, \frac{2c-2a}{2c-a-4b})| \leq 1 \text{ for } j = 1, 2.$$

It follows that, for $j = 1, 2$,

$$\begin{aligned} 1 &\geq |Q_j(\frac{1}{2}, -1)| \\ &= |\frac{1}{4}(a \pm \epsilon) + (b \pm \delta) - \frac{1}{2}(c \pm \gamma)| \\ &= |(\frac{1}{4}a + b - \frac{1}{2}c) \pm (\frac{\epsilon}{4} + \delta - \frac{\gamma}{2})| \\ &= |-1 \pm (\frac{\epsilon}{4} + \delta - \frac{\gamma}{2})| \\ &= 1 + |\frac{\epsilon}{4} + \delta - \frac{\gamma}{2}|, \end{aligned}$$

which shows that

$$\gamma = 2\delta + \frac{\epsilon}{2}.$$

From the fact that

$$|Q_j(\frac{c-4b}{2c-a-4b}, \frac{2c-2a}{2c-a-4b})| \leq 1 \quad (j = 1, 2)$$

we deduce that

$$(\dagger) \quad \delta = \frac{4b-c}{4(c-a)}\epsilon = (\frac{1}{4} - \frac{1}{2\sqrt{1-a}})\epsilon, \gamma = \frac{4b-a}{2(c-a)}\epsilon = (1 - \frac{1}{\sqrt{1-a}})\epsilon.$$

Hence,

$$\epsilon \leq 0, \delta \geq 0.$$

We claim that

$$c - \gamma > 1.$$

Otherwise. Then, $c - \gamma \leq 1$. By (†), it follows that

$$\begin{aligned} & c - \gamma \leq 1 \\ \Rightarrow\ & (a-1)\sqrt{1-a} + 2(1-a) \leq (\sqrt{1-a} - 1)\epsilon = (1 - \sqrt{1-a})|\epsilon| \\ \Rightarrow\ & (\sharp) \quad |\epsilon| \geq \frac{(1-a)(2-\sqrt{1-a})}{1-\sqrt{1-a}}. \end{aligned}$$

Since

$$a + |\epsilon| \leq \|Q_2\| = 1,$$

we have

$$|\epsilon| \leq 1 - a.$$

By (♯), we have

$$\frac{(1-a)(2-\sqrt{1-a})}{1-\sqrt{1-a}} \leq |\epsilon| \leq 1-a,$$

which is impossible because

$$\frac{(1-a)(2-\sqrt{1-a})}{1-\sqrt{1-a}} > 1-a.$$

Therefore, we have $c-\gamma > 1$. We will show that $\epsilon = 0$. By Theorem 2.1 (case 2), we have

$$\begin{aligned}
1 = \|Q_2\| &\geq \frac{(c - \frac{4b-a}{2(c-a)}\epsilon)^2 - 4(a-\epsilon)(b - \frac{4b-c}{4(c-a)}\epsilon)}{2(c - \frac{4b-a}{2(c-a)}\epsilon) - (a-\epsilon) - 4(b - \frac{4b-c}{4(c-a)}\epsilon)}, \\
&= \frac{(c^2-4ab) + (\gamma^2 - 2c\gamma + 4a\delta + 4b\epsilon - 4\epsilon\delta)}{(2c-a-4b) + (\epsilon + 4\delta - 2\gamma)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4 + (\gamma^2 - 2c\gamma + 4a\delta + 4b\epsilon - 4\epsilon\delta)}{4} \\
&= 1 + \frac{1}{4}\Big(\frac{-c(4b-a) + a(4b-c) + 4b(c-a)}{c-a}\epsilon + \frac{(2c-a-4b)^2}{4(c-a)^2}\epsilon^2\Big) \text{ (by †)} \\
&= 1 + \frac{(2c-a-4b)^2}{16(c-a)^2}\epsilon^2,
\end{aligned}$$

which shows that

$$\frac{(2c-a-4b)^2}{(c-a)^2}\epsilon^2 \leq 0,$$

hence, $\epsilon = 0$. Therefore, $0 = \epsilon = \delta = \gamma$. Thus we prove the claim.

Suppose that

$$0 < b < 1.$$

In this case we will derive a contradiction. First, assume that $c < a$. If $a \leq 4b$, then, by Theorem 2.1 (case 1),

$$1 = \|P\| = \max\{a, \frac{1}{4}a + b + \frac{1}{2}c, \frac{4ab-c^2}{4a}, \frac{4ab-c^2}{2c+a+4b}, \frac{4ab-c^2}{-2c+a+4b}\}.$$

Suppose that

$$1 = \frac{1}{4}a + b + \frac{1}{2}c.$$

If $c = 0$, then

$$\frac{-c^2+4ab}{4a} < 1, \frac{-c^2+4ab}{-2c+a+4b} < 1, \frac{-c^2+4ab}{2c+a+4b} < 1.$$

Note that

$$0 < a < 1, 0 < b < 1.$$

We claim that such P is not extreme. Indeed, let $n \in \mathbb{N}$ be such that

$$0 < a - \frac{1}{n} < a + \frac{1}{n} < 1, 0 < b - \frac{1}{4n} < b + \frac{1}{4n} < 1.$$

Let

$$Q_1(x,y) = (a + \frac{1}{n})x^2 + (b - \frac{1}{4n})y^2$$

and

$$Q_2(x,y) = (a - \frac{1}{n})x^2 + (b + \frac{1}{4n})y^2.$$

By Theorem 2.1 (case 1), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Hence

$$0 < c < 2.$$

Note that

$$\frac{-c^2+4ab}{4a} < 1, \ \frac{4ab-c^2}{2c+a+4b} < 1.$$

Since P is extreme, we claim that

$$1 = \frac{-c^2+4ab}{-2c+a+4b}.$$

Assume that

$$\frac{-c^2+4ab}{-2c+a+4b} < 1.$$

Let $n \in \mathbb{N}$ be such that

$$0 < c - \frac{1}{n} < c + \frac{1}{n} < a - \frac{1}{n} < a + \frac{1}{n} < 1, 0 < b - \frac{1}{4n} < b + \frac{1}{4n} < 1,$$
$$\frac{4(a \pm \frac{1}{n})(b \pm \frac{1}{4n}) - (c \mp \frac{1}{n})^2}{4(a \pm \frac{1}{n})} < 1, \ \frac{4(a \pm \frac{1}{n})(b \pm \frac{1}{4n}) - (c \mp \frac{1}{n})^2}{2(c \mp \frac{1}{n}) + (a \pm \frac{1}{n}) + 4(b \pm \frac{1}{4n})} < 1,$$

$$\frac{4(a\pm\frac{1}{n})(b\pm\frac{1}{4n})-(c\mp\frac{1}{n})^2}{-2(c\mp\frac{1}{n})+(a\pm\frac{1}{n})+4(b\pm\frac{1}{4n})}<1.$$

Let

$$Q_1(x,y)=(a+\frac{1}{n})x^2+(b+\frac{1}{4n})y^2+(c-\frac{1}{n})xy$$

and

$$Q_2(x,y)=(a-\frac{1}{n})x^2+(b-\frac{1}{4n})y^2+(c+\frac{1}{n})xy.$$

By Theorem 2.1 (case 1), $\|Q_j\|=1$ and $P=\frac{1}{2}(Q_1+Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Hence,

$$c=2-a>a,$$

which is a contradiction. Hence,

$$\frac{1}{4}a+b+\frac{1}{2}c<1,$$

which is also contradiction because

$$1=\|P\|=\max\{a,b,\frac{1}{4}a+b+\frac{1}{2}c\}<1.$$

Therefore, we should have

$$a>4b \ \text{ and } a>0.$$

By Theorem 2.1 (case 1),

$$1=\|P\|=\max\{a,b,\frac{1}{4}a+b+\frac{1}{2}c,\frac{|c^2-4ab|}{4a}\}.$$

Note that

$$\frac{|c^2-4ab|}{4a}<1.$$

Indeed,

$$\frac{|c^2-4ab|}{4a}\le\frac{c^2+4ab}{4a}<\frac{2a^2}{4a}=\frac{a}{2}<\frac{1}{2}.$$

Hence,

$$1=\frac{1}{4}a+b+\frac{1}{2}c.$$

In this case we claim that such P is not extreme. Indeed, let $n \in \mathbb{N}$ be such that

$$0 < c - \frac{1}{n} < c + \frac{1}{n} < a - \frac{1}{n} < a + \frac{1}{n} < 1, 0 < b - \frac{1}{4n} < b + \frac{1}{4n} < 1,$$
$$\frac{|4(a \pm \frac{1}{n})(b \pm \frac{1}{4n}) - (c \mp \frac{1}{n})^2|}{4(a \pm \frac{1}{n})} < 1.$$

Let

$$Q_1(x, y) = (a + \frac{1}{n})x^2 + (b + \frac{1}{4n})y^2 + (c - \frac{1}{n})xy$$

and

$$Q_2(x, y) = (a - \frac{1}{n})x^2 + (b - \frac{1}{4n})y^2 + (c + \frac{1}{n})xy.$$

By Theorem 2.1 (case 1), $\|Q_j\| = 1$ and $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that P is not extreme, reaching thus a contradiction. Therefore,

$$c \geq a.$$

If $a > 4b$, by Theorem 2.1 (case 2),

$$1 = \|P\| = \max\{a, b, \frac{1}{4}a + b + \frac{1}{2}c, \frac{c^2 - 4ab}{2c - a - 4b}\}.$$

Since P is extreme, by a similar argument to the one that allowed us to prove the equation of $(***)$,

$$1 = \frac{1}{4}a + b + \frac{1}{2}c = \frac{c^2 - 4ab}{2c - a - 4b}.$$

By a calculation, we have

$$c = 2 - 4b, a = 4b > 4b,$$

which contradicts the assumption that $a > 4b$. Hence,

$$a \leq 4b.$$

By Theorem 2.1 (case 2),

$$1 = \|P\| = \max\{a, b, \frac{1}{4}a + b + \frac{1}{2}c, \frac{|c^2 - 4ab|}{2c + a + 4b}\}.$$

Note that

$$\frac{|c^2-4ab|}{2c+a+4b}<1.$$

Indeed,

$$\begin{aligned}\frac{|c^2-4ab|}{2c+a+4b} &\leq \max\{\frac{c^2}{2c+a+4b},\frac{4ab}{2c+a+4b}\}<\max\{\frac{c^2}{2c},\frac{4ab}{4b}\}\\ &=\max\{\frac{c}{2},a\}<1.\end{aligned}$$

Hence,

$$1=\frac{1}{4}a+b+\frac{1}{2}c.$$

In this case we claim that such P is not extreme. Indeed, let $n\in\mathbb{N}$ be such that

$$\begin{aligned}&0<a-\frac{1}{n}<a+\frac{1}{n}<c-\frac{1}{n}<c+\frac{1}{n}<2, 0<b-\frac{1}{4n}<b+\frac{1}{4n}<1,\\ &\frac{4(a\pm\frac{1}{n})(b\pm\frac{1}{4n})-(c\mp\frac{1}{n})^2}{2(c\mp\frac{1}{n})+(a\pm\frac{1}{n})+4(b\pm\frac{1}{4n})}<1.\end{aligned}$$

Let

$$Q_1(x,y)=(a+\frac{1}{n})x^2+(b+\frac{1}{4n})y^2+(c-\frac{1}{n})xy$$

and

$$Q_2(x,y)=(a-\frac{1}{n})x^2+(b-\frac{1}{4n})y^2+(c+\frac{1}{n})xy.$$

By Theorem 2.1 (case 2), $\|Q_j\|=1$ and $P=\frac{1}{2}(Q_1+Q_2)$, which implies that P is not extreme, reaching thus a contradiction.Therefore, we complete the proof. $\square$

3. Applications to the polarization and unconditional constants of $\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$

In [22], Kim explicitly calculate $c_{pol}(2:d_*(1,w)^2)$ and $c_{unc}(2:d_*(1,w)^2)$ as follows:

(a) If $w \le \sqrt{2}-1$, then $c_{\rm pol}(2 : d_*(1,w)^2) = \frac{2(1+w^2)}{(1+w)^2}$;
(b) If $w > \sqrt{2}-1$, then $c_{\rm pol}(2 : d_*(1,w)^2) = 1+w^2$;
(c) If $w \le \sqrt{2}-1$, then $c_{\rm unc}(2 : d_*(1,w)^2) = \frac{1+w^2+\sqrt{2(1+w^4)}}{(1+w)^2}$;
(d) If $w > \sqrt{2}-1$, then $c_{\rm unc}(2 : d_*(1,w)^2) = \frac{1+w^2+\sqrt{(1+w^2)^2+4w^2}}{2}$.

Theorem 3.1. *Let $f \in \mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})^*$ and set $\alpha = f(x^2), \beta = f(y^2), \gamma = f(xy)$.*

$$\begin{aligned} \textit{Then, } \|f\| = \max\{&|\beta|, |\alpha+\frac{1}{4}\beta|+|\gamma|, \\ &|\alpha+\frac{3}{4}\beta|, |\alpha+(\frac{c^2}{4}-1)\beta|+c|\gamma| \ (0\le c\le 1), \\ &|a\alpha+(\frac{a+4\sqrt{1-a}}{4}-1)\beta|+(a+2\sqrt{1-a})|\gamma| \ (0\le a\le 1)\}. \end{aligned}$$

Proof. It follows from Theorem 2.2 and the fact that
$\|f\| = \sup_{P\in ext_{B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}}} |f(P)|$. $\square$

Note that if $\|f\| = 1$, then $|\alpha| \le 1, |\beta| \le 1, |\gamma| \le \frac{1}{2}$.

Theorem 3.2. *([24]) Let $T((x_1,y_1),\ (x_2,y_2)) := (a,b,c) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(\frac{1}{2})})$.*

$$\textit{Then, } \|T\| = \max\{|a|, \frac{1}{2}|a|+|c|, |\frac{1}{4}a-b|, |\frac{1}{4}a+b|+|c|\}.$$

Theorem 3.3. *(a) $c_{pol}(2 : \mathbb{R}^2_{h(\frac{1}{2})}) = \frac{5}{4}$;*
(b) $c_{unc}(2 : \mathbb{R}^2_{h(\frac{1}{2})}) = \frac{3}{2}$.

Proof. Let

$$\begin{aligned} P_1(x,y) &= y^2, \\ P_2(x,y) &= x^2+\frac{1}{4}y^2 \pm xy, \\ P_3(x,y) &= x^2+\frac{3}{4}y^2, \\ P_{4,c}(x,y) &= x^2+(\frac{c^2}{4}-1)y^2 \pm cxy \ (0\le c\le 1), \\ P_{5,a}(x,y) &= ax^2+(\frac{a+4\sqrt{1-a}}{4}-1)y^2 \pm (a+2\sqrt{1-a})xy \ (0\le a\le 1). \end{aligned}$$

(a): Note that

$$\begin{aligned}
\check{P}_1((x_1,y_1),(x_2,y_2)) &= y_1y_2,\\
\check{P}_2((x_1,y_1),(x_2,y_2)) &= x_1x_2+\frac{1}{4}y_1y_2 \pm \frac{1}{2}(x_1y_2+x_2y_1),\\
\check{P}_3((x_1,y_1),(x_2,y_2)) &= x_1x_2+\frac{3}{4}y_1y_2,\\
\check{P}_{4,c}((x_1,y_1),(x_2,y_2)) &= x_1x_2+(\frac{c^2}{4}-1)y_1y_2 \pm \frac{c}{2}(x_1y_2+x_2y_1)\ (0\le c\le 1),\\
\check{P}_{5,a}((x_1,y_1),(x_2,y_2)) &= ax_1x_2+(\frac{a+4\sqrt{1-a}}{4}-1)y_1y_2\\
&\quad \pm \frac{a+2\sqrt{1-a}}{2}(x_1y_2+x_2y_1) \quad (0\le a\le 1).
\end{aligned}$$

Note that, by Theorem 3.2,

$$\|\check{P}_1\| = 1 = \|\check{P}_2\| = \|\check{P}_3\|.$$

Claim: $\|\check{P}_{4,c}\| = \frac{5-c^2}{4}$ for $0\le c\le 1$ and $\|\check{P}_{5,a}\| = a+\sqrt{1-a}$ for $0\le a\le 1$.
By Theorem 3.2,

$$\|\check{P}_{4,c}\| = \max\{1, \frac{1+c}{2}, \frac{5-c^2}{4}, \frac{-c^2+2c+3}{4}\} = \frac{5-c^2}{4}.$$

Hence,

$$\sup_{0\le c\le 1}\|\check{P}_{4,c}\| = \sup_{0\le c\le 1}\frac{5-c^2}{4} = \frac{5}{4}.$$

By Theorem 3.2,

$$\|\check{P}_{5,a}\| = \max\{a, a+\sqrt{1-a}, 1-\sqrt{1-a}, a-1+2\sqrt{1-a}\} = a+\sqrt{1-a}.$$

Hence,

$$\sup_{0\le a\le 1}\|\check{P}_{5,a}\| = \sup_{0\le a\le 1} a+\sqrt{1-a} = \frac{5}{4} \text{ at } \ a=\frac{3}{4}.$$

By the Krein-Milman Theorem,

$$c_{\mathrm{pol}}(2:\mathbb{R}^2_{h(\frac{1}{2})}) = \sup\{\|\check{P}_1\|, \|\check{P}_2\|, \|\check{P}_3\|, \|\check{P}_{4,c}\|, \|\check{P}_{5,a}\| : 0\le c, a\le 1\} = \frac{5}{4}.$$

(b): Note that

$$|P_1|(x,y) = y^2,$$

$$|P_2|(x,y) = x^2 + \frac{1}{4}y^2 + xy,$$
$$|P_3|(x,y) = x^2 + \frac{3}{4}y^2,$$
$$|P_{4,c}|(x,y) = x^2 + (1 - \frac{c^2}{4})y^2 + cxy \ (0 \le c \le 1),$$
$$|P_{5,a}|(x,y) = ax^2 + (1 - \frac{a + 4\sqrt{1-a}}{4})y^2 + (a + 2\sqrt{1-a})xy \ (0 \le a \le 1).$$

Note that, by Theorem 2.1,

$$|||P_1||| = 1 = |||P_3|||, |||P_2||| = \frac{3}{2}.$$

Claim: $|||P_{4,c}||| = \max\{\frac{-c^2+2c+5}{4}, \frac{4-2c^2}{-c^2-2c+5}\}$ for $0 \le c \le 1$ and $|||P_{5,a}||| = 1 + \frac{a}{2}$ for $0 \le a \le 1$.

By Theorem 2.1 (case 1),

$$|||P_{4,c}||| = \max\{\frac{-c^2+2c+5}{4}, \frac{4-2c^2}{-c^2-2c+5}\} = \frac{-c^2+2c+5}{4} \quad \text{for } 0 \le c < 1.$$

For $c = 1$, by Theorem 2.1 (case 2),

$$|||P_{4,1}||| = \frac{3}{2}.$$

Hence,

$$\sup_{0 \le c \le 1} |||P_{4,c}||| = \sup_{0 \le c \le 1} \frac{-c^2+2c+5}{4} = \frac{3}{2}.$$

By Theorem 2.1,

$$|||P_{5,a}||| = \max\{\frac{2+a}{2}, \frac{a^2 - 4a + 2 + 4a\sqrt{1-a}}{2+a}\} = \frac{2+a}{2} \quad \text{for } \ 0 \le a \le 1.$$

Hence,

$$\sup_{0 \le a \le 1} |||P_{5,a}||| = \sup_{0 \le c \le 1} \frac{2+a}{2} = \frac{3}{2}.$$

By the Krein-Milman Theorem,

$$c_{\text{unc}}(2 : \mathbb{R}^2_{h(\frac{1}{2})}) = \sup\{|||P_1|||, |||P_2|||, |||P_3|||, |||P_{4,c}|||, |||P_{5,a}||| : 0 \le c, a \le 1\}$$

$$= \frac{3}{2}.$$

We complete the proof. $\square$

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